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# A LITTLE CHARITY GUARANTEES ALMOST ENVY-FREENESS*

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**Abstract.** The fair division of indivisible goods is a very well-studied problem. The goal of this problem is to distribute $m$ goods to $n$ agents in a "fair" manner, where every agent has a valuation for each subset of goods. We assume monotone valuations. Envy-freeness is the most extensively studied notion of fairness. However, envy-free allocations do not always exist when goods are indivisible. The notion of fairness we consider here is "envy-freeness up to any good," EFX, where no agent envies another agent after the removal of any single good from the other agent's bundle. It is not known if such an allocation always exists. We show there is always a partition of the set of goods into $n+1$ subsets $(X_1, \ldots, X_n, P)$, where for $i \in [n]$, $X_i$ is the bundle allocated to agent $i$ and the set $P$ is unallocated (or donated to charity) such that we have (1) envy-freeness up to any good, (2) no agent values $P$ higher than her own bundle, and (3) fewer than $n$ goods go to charity, i.e., $|P| < n$ (typically $m \gg n$). Our proof is constructive and leads to a pseudopolynomial time algorithm to find such an allocation. When agents have additive valuations and $|P|$ is large (i.e., when $|P|$ is close to $n$), our allocation also has a good maximin share (MMS) guarantee. Moreover, a minor variant of our algorithm also shows the existence of an allocation that is 4/7 *groupwise* maximin share (GMMS): this is a notion of fairness stronger than MMS. This improves upon the current best bound of 1/2 known for an approximate GMMS allocation. (Very recently and independently, Amanatidis, Ntokos, and Markakis [*Theoret. Comput. Sci.*, 841 (2020), pp. 94–109], also showed the existence of a 4/7-GMMS allocation.)

**Key words.** fair division, EFX allocations, MMS allocations

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## 1. Introduction.

Fair division of goods among competing agents is a fundamental problem in Economics and Computer Science. There is a set $M$ of $m$ goods and the goal is to *allocate* goods among $n$ agents in a *fair* way. An allocation is a partition of $M$ into disjoint subsets $X_1, \ldots, X_n$ where $X_i$ is the set of goods given to agent $i$. When can an allocation be considered "fair"? One of the most well-studied notions of fairness is *Envy-freeness*. Every agent has a *value* associated with each subset of $M$ and agent $i$ *envies* agent $j$ if $i$ values $X_j$ more than $X_i$. An allocation is *envy-free* if no agent envies another. An envy-free allocation can be regarded as a fair and desirable partition of $M$ among the $n$ agents since no agent envies another; as mentioned in [30], such a mechanism of partitioning land dates back to the Bible.

Unlike land which is divisible, goods in our setting are *indivisible* and an envy-free allocation of the given set of goods need not exist. Consider the following simple example with two agents and a single good that both agents desire: one of the agents has to receive this good and the other agent envies her. Since envy-free allocations need not exist, several relaxations have been considered.

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**Relaxations.** Budish [13] introduced the notion of *EF*1: this is an allocation of goods that is "envy-free up to *one* good." In an EF1 allocation, agent $i$ may envy agent $j$, but this envy would vanish as soon as *some* good is removed from $X_j$. Note that no good is really removed from $X_j$: this is simply a way of assessing how much $i$ values $X_j$ more than $X_i$. That is, if $i$ values $X_j$ more than $X_i$, then there exists some $g \in X_j$ such that $i$ values $X_i$ at least as much as $X_j \setminus \{g\}$. Going back to the example of two agents and a single good, the allocation where one agent receives this good is EF1. It is known that EF1 allocations always exist; as shown by Lipton et al. [29], such an allocation can be efficiently computed.[1]

Caragiannis et al. [15] introduced a notion of envy-freeness called *EFX* that is stronger than EF1. An EFX allocation is one that is "envy-free up to *any* good." In an EFX allocation, agent $i$ may envy agent $j$, but this envy would vanish as soon as *any* good is removed from $X_j$. Thus every EFX allocation is also EF1 but not every EF1 allocation is EFX.

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| Agent 1 | 1 | 1 | 3 |
| Agent 2 | 1 | 1 | 3 |

Consider the simple example given above: there are three goods $a, b, c$ and two agents with *additive valuations*[2] (formally defined in section 1.1) as described below. Both agents value $c$ three times as much as $a$ or $b$. Observe that there is no envy-free allocation here. The allocation where agent 1 gets $\{a\}$ and agent 2 gets $\{b, c\}$ is EF1 but not EFX. On the other hand, the allocation where agent 1 gets $\{a, b\}$ and agent 2 gets $\{c\}$ is EFX. Indeed, the latter allocation seems fairer than the former allocation. As said in [14], "Arguably, EFX is the best fairness analog of envy-freeness of indivisible items." While it is known that EF1 allocations always exist, the question of whether EFX allocations always exist is still an open problem (despite "significant effort," according to [15]). In fact, Procaccia [31] calls this question the current biggest open problem in fair division:

> "This fundamental and deceptively accessible question is open. In my view, it is the successor of envy-free cake cutting as fair division's biggest problem."

Plaut and Roughgarden [30] showed that EFX allocations always exist (i) when there are only two agents or (ii) when all $n$ agents have the same valuation function. Moreover, it was shown in [30] that exponentially many value queries may be needed to determine EFX allocations even in the restricted setting where there are only two agents with identical *submodular* valuation functions.[3] It was not known until very recently if an EFX allocation always exists even when there are only three agents with additive valuations. A positive answer to this question was very recently given by Chaudhury, Garg, and Mehlhorn [17]. This was said in [30]: "We suspect that at least for general valuations, there exist instances where no EFX allocation exists." Note that a general valuation is one that is *monotone* (defined in section 1.1.1).

**A relaxation of EFX.** Very recently, Caragiannis, Gravin, and Huang [14] introduced a more relaxed notion of EFX called *EFX-with-charity.* This is a partial

[1] The algorithm in [29] was published in 2004 with a different property and EF1 was proposed in 2011.

[2] The value of a set $S \subseteq M$ is the sum of values of goods in $S$.

[3] These are valuation functions with decreasing marginal values.

allocation that is EFX, i.e., the entire set of goods need not be distributed among the agents. So some goods may be left *unallocated* and it is assumed that these unallocated goods are donated to charity. There is a very simple allocation that is EFX-with-charity where *no* good is assigned to any agent—thus all goods are donated to charity. Obviously, this is not an interesting allocation and one seeks allocations with better guarantees on the allocated and unallocated goods. One such allocation was shown in [14].

Let $X^* = \langle X_1^*, \ldots, X_n^* \rangle$ be an optimal *Nash social welfare* allocation on the entire set of goods, i.e., an allocation that maximizes $\Pi_{i=1}^n v_i(X_i^*)$, where $v_i$ is agent $i$'s valuation function. It was shown in [14] that when agents have additive valuations, there always exists an EFX-with-charity allocation $X = \langle X_1, \ldots, X_n \rangle$ where every agent receives at least half the value of her allocation in $X^*$. Interestingly, as shown in [14], $X_i \subseteq X_i^*$ for all $i$. Unfortunately, there are no upper bounds on the *number* of unallocated goods or on the *value* any agent has for the set of goods donated to charity.

We believe these are important questions to ask. The ideal allocation is one that is EFX and allocates *all* goods, so we would like a guarantee that a large number of goods have been allocated to agents. Moreover, since EFX allocations are relaxations of envy-free allocations, it is in the same spirit that we seek an EFX (partial) allocation where nobody envies the set of unallocated goods. The allocation in [14] gives no guarantee either on the number of unallocated goods or on whether any agent values the set of unallocated goods more than her own bundle. Here we consider the notion of *EFX-with-bounded-charity*. That is, we seek EFX-with-charity allocations with bounds on the set given to charity, i.e., a bound on the *size* and a bound on the *value* of the set of goods donated to charity.

**Efficient allocations.** There has also been a significant amount of work done in determining fair allocations that are *efficient*. Efficiency is usually a measure of the overall welfare that an allocation achieves while still being fair. A very common measure of efficiency is *Pareto-optimality*: an allocation $X$ is said to be Pareto-optimal if there is no allocation $Y$ such that $v_i(Y_i) \geq v_i(X_i)$ for all agents $i$ and $v_i(Y_j) > v_i(X_j)$ for some agent $j$. Intuitively, we cannot make any agent happier without reducing the valuation of some other agent. With this is in mind, a natural question to ask is the following: can we get fair allocations (EF1 or EFX) that are also Pareto-optimal?

Caragiannis et al. [15] answered the above question positively for EF1, by showing that an optimal Nash social welfare allocation is EF1 (fair) and Pareto-optimal (efficient). Thereafter, Nash social welfare in itself has been used as a measure of efficiency [14].[4] Furthermore, Plaut and Roughgarden [30] showed that there exists an instance where no EFX allocation is Pareto-optimal. Thus, the question we wish to answer is: can we get good relaxations of EFX with high Nash social welfare? It turns out that by modifying the first step of our algorithm, we can determine an EFX allocation with bounded charity (fair) that has high Nash social welfare (efficient) (see subsection 1.1.3 for a more precise and detailed statement).

### 1.1. Our results.

Let $N = [n]$ be the set of agents. Every agent $i \in [n]$ has a valuation function $v_i : 2^M \to \mathbb{R}_{\geq 0}$, where $M$ is the set of $m$ goods.

#### 1.1.1. Monotone valuations.

We show our main existence result for *monotone* valuation functions, i.e., the only assumption we make on any valuation function $v_i$ is that it is monotone, so $S \subseteq T$ implies $v_i(S) \leq v_i(T)$.

[4]The higher the Nash social welfare of an allocation, the more efficient the allocation is.

In contrast, the EFX-with-charity allocation in [14] works only for *additive valuations*, i.e., for any $S \subseteq M$ and $i \in [n]$, we have $v_i(S) = \sum_{g \in S} v_i(\{g\})$.

We show there always exists an allocation[5] $X = \langle X_1, \ldots, X_n \rangle$ that satisfies the following properties:

1. $X$ is EFX, i.e., for any two agents $i, j$, $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for any $g \in X_j$;
2. $v_i(X_i) \geq v_i(P)$ for all agents $i$, where $P = M \setminus \cup_{i=1}^{n} X_i$ is the set of unallocated goods;
3. $|P| < n$ (recall that $n$ is the number of agents and typically $n \ll m$).

Note that our result implies the following simple observation—among the $n$ agents, if there is just *one* agent who has no preferences (say, for agent $i$, we have $v_i(S) = 0$ for all $S \subseteq M$), then a complete EFX allocation always exists for monotone valuations: Find an EFX allocation satisfying all the three properties mentioned above, on the set of agents $N \setminus \{i\}$. The allocation among the agents in $N \setminus \{i\}$ is EFX (by condition 1) and we now allocate $P$ (the set of unallocated goods) to agent $i$. Observe that agent $i$ envies nobody as $i$ has the same value for every subset of goods and nobody envies $i$ as nobody envies $P$ (by condition 2).

Our proof is constructive. We start with no goods being allocated to the agents and find the claimed allocation by at most $nmV/\Delta + m$ applications of three simple update rules, where $n$ is the number of agents, $m$ is the number of goods, $V = \max_i v_i(M)$ is the maximum valuation of any agent, and $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$ is the minimum difference between distinct valuations.

The update rules use a *minimal-envied-subset*-oracle: given $S \subseteq M$ such that there is an agent who envies $S$, find a subset $Z \subseteq S$ such that there is an agent who envies $Z$ and no agent envies a proper subset of $Z$. This oracle can be realized by a simple algorithm that uses at most $nm$ value queries. Thus we show that for monotone valuations, an EFX allocation that satisfies properties 1–3 can be computed with $\text{poly}(n, m, V, 1/\Delta)$ value queries, i.e., in pseudopolynomial time. With a slight modification of our pseudopolynomial time algorithm, we can determine a $(1-\varepsilon)$-EFX allocation with bounded charity[6] in $\text{poly}(n, m, \log(V/\Delta), 1/\varepsilon)$ time.

#### 1.1.2. Additive valuations.

The most well-understood class of valuation functions is the set of *additive* valuations. We consider the case when all agents have additive valuations and show that our allocation or very minor variants of our allocation can guarantee several other notions of fairness.

**Number of unallocated goods and MMS guarantee.** Another interesting and well-studied notion of fairness is *maximin share* (MMS). Suppose agent $i$ has to partition $M$ into $n$ *bundles* (or sets) knowing that she would receive the worst bundle with respect to her valuation. Then $i$ will choose a partition of $M$ that maximizes the valuation of the worst bundle (with respect to her valuation). The value of this worst bundle is the MMS of agent $i$. An important question here is, does there always exist an allocation of $M$ where every agent gets a bundle worth at least her MMS?

[5] Henceforth, allocations are *partial* and we will use "complete allocation" to refer to one where all goods are allocated.

[6] $X$ is a $(1-\varepsilon)$-EFX allocation with bounded charity, when we have the following:
1. For any pair of agents $i, j$, $v_i(X_i) \geq (1-\varepsilon) \cdot v_i(X_j \setminus \{g\})$ for any $g \in X_j$;
2. $v_i(X_i) \geq (1-\varepsilon) \cdot v_i(P)$ for all agents $i$; and
3. $|P| < n$.

Formally, let $N$ and $M$ be the sets of $n$ agents and $m$ goods, respectively. Using $\mathcal{X}$ to denote the set of all complete allocations, we define the MMS of an agent (say, $i$) as follows:

$$MMS_i(n, M) = \max_{\langle X_1,\ldots,X_n\rangle \in \mathcal{X}} \min_{j\in[n]} v_i(X_j).$$

The goal is to determine an allocation $\langle X_1, X_2, \ldots, X_n\rangle$ of $M$ such that for every $i$, we have $v_i(X_i) \geq MMS_i(n, M)$. This question was first posed by Budish [13]. Kurokawa, Procaccia, and Wang [28] showed that such an allocation need not exist, even in the restricted setting of only three agents! Thereafter, *approximate-MMS* allocations were studied [28, 23, 26, 25] and there are polynomial time algorithms to find allocations where for all $i$, agent $i$ gets a bundle of value at least $\alpha \cdot MMS_i(n, M)$; the best guarantee known for $\alpha$ was $3/4 - \epsilon$ by Ghodsi et al. [26] (for any $\epsilon > 0$) and this was very recently improved to $3/4$ by Garg and Taki [25].

Amanatidis, Birmpas, and Markakis [1] showed that any complete EFX allocation is also a $\frac{4}{7}$-MMS allocation. We show that our allocation promises better MMS guarantees when the number of unallocated goods is large. Let $X = \langle X_1, \ldots, X_n\rangle$ be our allocation as described by properties 1–3 above and $P$ be the set of unallocated goods. For any agent $i \in [n]$, we have

$$v_i(X_i) \;\geq\; \frac{1}{2 - |P|/n} MMS_i(n, M).$$

Hence, the larger the number of unallocated goods, the better are the guarantees that we get on MMS. The extreme values are $|P| = 0$ and $|P| = n-1$. When $|P| = 0$, we have a complete EFX allocation and when $|P| = n-1$, we have an EFX allocation that is an *almost-MMS* allocation: $v_i(X_i) \geq (1 - 1/n) \cdot MMS_i(n, M)$ for all $i$.

**Improved guarantees for groupwise MMS.** Barman et al. [7] recently introduced a notion of fairness called *groupwise maximin share* (GMMS), which is stronger than MMS. An allocation is said to be GMMS if the MMS condition is satisfied for *every subgroup* of agents and the union of the sets of goods allocated to them. Formally, a complete allocation $X = \langle X_1, X_2, \ldots, X_n\rangle$ is $\alpha$-GMMS if for any $N' \subseteq N$ and all $i \in N'$, we have $v_i(X_i) \geq \alpha \cdot MMS_i(n', \bigcup_{j\in N'} X_j)$, where $n' = |N'|$. Every GMMS allocation, i.e., $\alpha = 1$, is also a complete EFX allocation [7].

It is known [7] that GMMS strictly generalizes MMS. In particular, it was shown that GMMS allocations rule out some very unsatisfactory allocations that have MMS guarantees. For example, consider an instance with $n$ agents with additive valuations and a set $M$ of $n-1$ goods and every agent has a valuation of one for each good. Since the number of goods is less than the number of agents, we have $MMS_i(n, M) = 0$ for every agent $i$. So any allocation is an MMS allocation. It is not hard to see that the only allocations that have a GMMS guarantee are those where $n - 1$ agents get one good each and one agent is left without any goods. See subsection 2.1 in [7] for more discussion. Naturally, it is a harder problem to approximate GMMS than MMS. While $\frac{3}{4}$-MMS allocations always exist, the largest $\alpha$ for which $\alpha$-GMMS allocations are known to exist is $\frac{1}{2}$ [7]. We extend the result of Amanatidis et al. [1] for MMS to show the following:

- A $\frac{4}{7}$-GMMS allocation always exists and it can be computed in pseudo-polynomial time.

In particular, we show that modifying the last step of our algorithm results in a complete allocation that is $\frac{4}{7}$-GMMS. As mentioned earlier, very recently and independently, Amanatidis, Ntokos, and Markakis [3] showed the same approximation.

**1.1.3. Efficiency of our allocation.** We remark that our algorithm is very "robust" in the sense that it computes an EFX allocation with bounded charity starting from any partial EFX allocation (trivially, we can start with the *empty* allocation, i.e., no good is assigned to any agent) and every agent in the final allocation has a valuation at least as high as her initial valuation (since we maintain the invariant that the valuation of any agent never decreases in the entire course of the algorithm). This robustness helps us to determine an EFX allocation $X = \langle X_1, \ldots, X_n \rangle$ with bounded charity that also has a high Nash social welfare, i.e., a high value of $(\Pi_{i=1}^n v_i(X_i))^{1/n}$.

**Additive valuations.** We show that modifying the starting step of our algorithm ensures that our EFX allocation $X$ with bounded charity also satisfies $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ as promised in [14], where $X^* = \langle X_1^*, \ldots, X_n^* \rangle$ is an optimal Nash social welfare allocation. Here we use the allocation computed in [14] as a black box in our starting step, and thus our result can be regarded as an extension of the result in [14].

**Subadditive valuations.** A valuation $v$ is subadditive if $v(S)+v(T) \geq v(S \cup T)$ for all $S, T \subseteq M$. It follows from recent work by Chaudhury, Garg, and Mehta [18] that a careful modification of the first step of our algorithm results in an EFX allocation with bounded charity that also achieves an $O(n)$ approximation of Nash social welfare[7] when agents have subadditive valuations. Barman et al. [6] showed that it requires an exponential number of value queries to provide a sublinear approximation to Nash social welfare under subadditive valuations. Thus, by choosing the initial partial EFX allocation carefully, for any fixed $\varepsilon > 0$, our algorithm yields a $(1-\varepsilon)$-EFX allocation with bounded charity, using a polynomial number of value queries and it achieves the best approximation of Nash social welfare that is possible with a polynomial number of value queries.

**1.2. Our techniques.** We now give an overview of the main ideas used to find our EFX allocation. We first recall the algorithm of Lipton et al. [29] for finding an EF1 allocation. They use the notion of an *envy-graph*: here each vertex corresponds to an agent and there is an edge $(i, j)$ if and only if $i$ envies $j$. The invariant maintained is that the envy-graph is a DAG: a cycle corresponds to a cycle of envy and by swapping bundles along a cycle, every agent in the cycle becomes better-off. More precisely, if $i_0 \to i_1 \to i_2 \to \cdots \to i_{\ell-1} \to i_0$ is a cycle in the envy-graph, then reassigning $X_{i_{j+1}}$ to agent $i_j$ for $0 \leq j < \ell$ (indices are to be read modulo $\ell$) will increase the valuation of every agent in the cycle. Thus cycles can be eliminated (see Figure 1). Also if there was an edge from $s$ to some $i_k$ where $s$ is not a part of the cycle, this edge just gets directed now from $s$ to $i_{k-1}$ after we exchange bundles along the cycle.

The algorithm in [29] runs in rounds and always maintains an allocation that is EF1. At the beginning of every round, an unenvied agent $s$ (this is a source vertex in this DAG) is identified and an unallocated good $g$ is allocated to $s$. The new allocation is EF1, as nobody will envy the bundle of $s$ after removing the good $g$.

**The reallocation operation.** We now highlight a key difference between an EF1 allocation and an EFX allocation. From the algorithm of Lipton et al. [29], it is clear that given an EF1 allocation on a set $M_0$ of goods, one can determine an EF1 allocation on $M_0 \cup M_1$, for any $M_1 \subseteq M \setminus M_0$, by simply adding goods in $M_1$ one-by-one to the existing bundles and changing the owners (if necessary) in a clever way.

[7] In fact, it achieves an $O(n)$ approximation of a much larger class of efficiency functions, namely, the generalized $p$-mean welfare function. Independently, Barman et al. [6] also showed how to achieve an $O(n)$ approximation for the generalized $p$-mean welfare (hence also Nash social welfare).

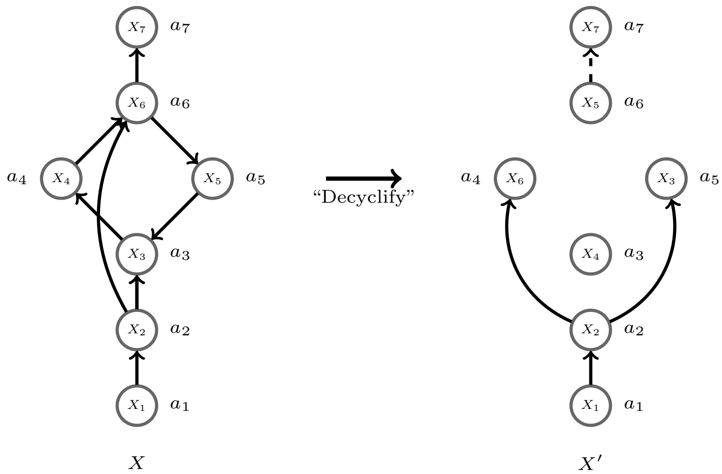


FIG. 1. *Eliminating cycles in the envy-graph. There is a cycle in allocation $X$: $a_3 \to a_4 \to a_6 \to a_5 \to a_3$. The bundles are exchanged along the cycle in allocation $X'$ and the agents along the cycle strictly improve their valuations. The envy edges outgoing from the agents in the cycle may or may not be there in $X'$ (the edge from $a_6$ to $a_7$), while the envy edges which were incoming to the agents in the cycle reorient themselves (instead of the edge from $a_2$ to $a_3$ (and from $a_2$ to $a_6$) in $X$, there is the edge from $a_2$ to $a_5$ (from $a_2$ to $a_4$) in $X'$).*

Intuitively, we never need to cut or merge the bundles formed in any EF1 allocation. We can just append the unallocated goods appropriately to the current bundles.

The above strategy is very far from true for EFX. Consider the example given below where there are three agents with additive valuations and four goods $a, b$, $c$, and $d$; let $\varepsilon \in (0, 1)$.

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| Agent 1 | $\varepsilon$ | 1 | 1 | 2 |
| Agent 2 | 1 | $\varepsilon$ | 1 | 2 |
| Agent 3 | 1 | 1 | $\varepsilon$ | 2 |

An EFX allocation for the first three goods has to give exactly one of $a, b, c$ to each of the three agents. However, an EFX allocation for all the four goods has to allocate the singleton set $\{d\}$ to some agent (say, agent 1) and, say, $\{a\}$ to agent 2 and $\{b, c\}$ to agent 3. Thus the allocation needs to be *cut and merged*. When there are many agents, each with her own valuation, figuring out the cut-and-merge operations is the difficult step. Here we implement our global reallocation operation as follows.

**Improving social welfare.** Suppose we have an EFX allocation

$$X = \langle X_1, \ldots, X_n \rangle$$

on some subset $M_0 \subset M$. We would now like to add a good $g \in M \setminus M_0$. However, we will not be able to guarantee an EFX allocation on $M_0 \cup \{g\}$. What we will ensure is that either case (i) or case (ii) occurs:

(i) We have an EFX allocation $X' = \langle X'_1, \ldots, X'_n \rangle$ on a subset of $M_0 \cup \{g\}$ such that $v_i(X'_i) \geq v_i(X_i)$ for all $i$ and for at least one agent $j$ we have $v_j(X'_j) > v_j(X_j)$. Thus $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$; in other words, the *social welfare* strictly improves.

(ii) We have an EFX allocation on $M_0 \cup \{g\}$ and the social welfare does not decrease.

Hence in each step of our algorithm, we either increase social welfare or we increase the number of allocated goods without decreasing social welfare—thus we always make progress. This is similar to the approach used by Plaut and Roughgarden [30] to guarantee the existence of $\frac{1}{2}$-EFX when agents have subadditive valuations. We now outline how we ensure that one of cases (i) and (ii) has to happen.

For simplicity of exposition, let us assume the envy-graph corresponding to our starting EFX allocation $X$ has a single source $s$. Add $g$ to $s$'s bundle: if nobody envies $s$ up to any single good, i.e., if $v_i(X_i) \geq v_i(X_s \cup \{g\} \setminus \{g'\})$ for all $i \in [n]$ and all $g' \in X_s \cup \{g\}$, then we are in an easy case as we have an EFX allocation on $M_0 \cup \{g\}$. In this case, we "decycle" the envy-graph (if cycles are created) and continue. Observe that swapping bundles along a cycle in the envy-graph increases social welfare.

**Most envious agent.** Assume now that there are one or more agents who envy $s$ up to some good after $g$ is allocated to $s$. To resolve this, we introduce the concept of *most envious agent*. If there is an agent who envies $s$ up to some good, then there is some $S \subset X_s \cup \{g\}$ that is *envied* (we use $\subset$ to denote a proper subset). That is, there is an agent who values $S$ more than her bundle. Let $Z$ be any inclusionwise minimal subset of $X_s \cup \{g\}$ that is envied. So there exists an agent who envies $Z$ and no proper subset of $Z$ is envied by any agent. Any agent who envies $Z$ is a most envious agent of $X_s \cup \{g\}$. Whenever there is an agent who envies $s$ up to some good, observe that there has to exist a most envious agent of $X_s \cup \{g\}$.

Let $Z$ be an inclusionwise minimal subset of $X_s \cup \{g\}$ that is envied and let $t$ be an agent such that $v_t(X_t) < v_t(Z)$. Recall the assumption that $s$ is the only source, so there is a path $s \to i_1 \to \cdots \to i_{k-1} \to t$ in the envy-graph. We do a leftwise shift of bundles along this path, so $s$ gets $i_1$'s bundle, and for $1 \leq r \leq k-1$, $i_r$ gets $i_{r+1}$'s bundle (where $i_k = t$), and finally $t$ gets $Z$. The goods in $X_s \cup \{g\} \setminus Z$ are thrown back into the pool of unallocated goods (see Figure 2).

Observe that every agent in this path is strictly better-off now than in the allocation $X$ and no other agent is worse-off. Moreover, since no proper subset of $Z$ is envied, there is no agent envying any other agent up to any single good. Thus we have a desired EFX allocation $X'$. When there are multiple sources, we can adapt this technique *provided* there are enough unallocated goods; in particular, the number of unallocated goods must be at least the number of sources in the envy-graph. We describe this in detail in section 2.

We would like to contrast the above approach with other EFX algorithms [30, 14]. The $\frac{1}{2}$-EFX algorithm by Plaut and Roughgarden [30] either merges $g$ (the new good) with an existing bundle or allocates the singleton set $\{g\}$ to an agent. The EFX-with-charity algorithm by Caragiannis, Gravin, and Huang [14] takes an allocation of maximum Nash social welfare as input and then permanently removes some goods from the instance. We regard the notion of "most envious agent" that shows a natural way of breaking up a bundle to preserve envy-freeness up to any good as one of the innovative contributions of our work.

**Our other results.** Regarding our result with approximate MMS guarantee, if the number of unallocated goods in our EFX allocation is large, then the number of sources also has to be large: these are unenvied agents. Moreover, no agent envies the set of unallocated goods. Suppose for now that $|P| = n - 1$. This means every agent is a source. So no agent envies the bundle of any other agent and also the

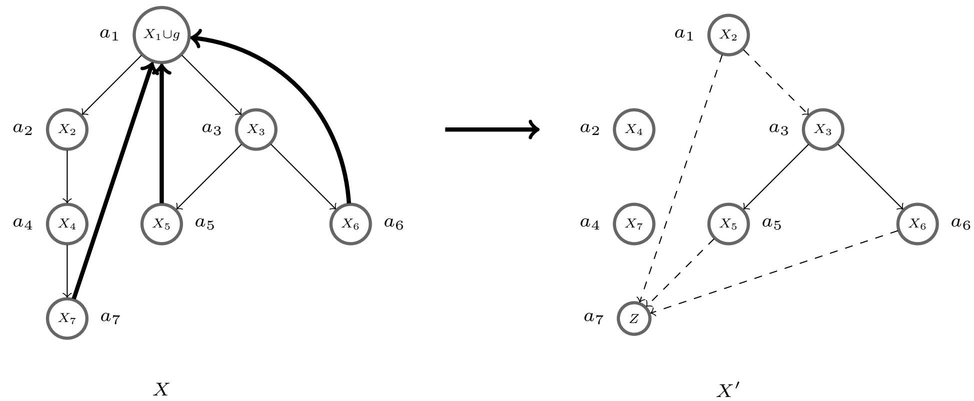


FIG. 2. *Illustration of the update rule. The figure on the left indicates the envy-graph corresponding to the allocation $X$. Thin edges indicate* weak envy *edges: a thin edge from $a_i$ to $a_j$ signifies that $a_i$ envies $a_j$, but $a_i$ does not envy any proper subset of $a_j$'s bundle. The thick edges indicate* strong envy *edges: a thick edge from $a_i$ to $a_j$ indicates that $a_i$ also envies some proper subset of $a_j$'s bundle. Initially the envy-graph has only thin edges with $a_1$ as the only source. Agents $a_7$, $a_5$, and $a_6$* strongly envy *$a_1$ when we give $a_1$ the good $g$ and $a_7$ is a most envious agent. The figure on the right indicates the envy-graph of the allocation $X'$ that we obtain after applying the update rule. We shift the bundles along the path $a_1 \to a_2 \to a_4 \to a_7$ and give $a_7$ a subset $Z$ of $X_s \cup \{g\}$ that she envies and no proper subset of $Z$ is envied by any agent. All the agents along the path $(a_1, a_2, a_4, a_7)$ strictly improve their valuations. All the thick edges disappear as no agent envies a proper subset of $Z$ (there may or may not be thin edges directed toward $a_7$, depending on whether $a_1$, $a_5$, or $a_6$ envy $Z$).*

set of unallocated goods. We assumed valuations to be additive here, so this means $v_i(X_i) \geq v_i(M)/(n+1)$ for each agent $i$. Thus we have

$$\begin{aligned} v_i(X_i) \; \geq \; \frac{v_i(M)}{n+1} \; &= \; (1+1/n)^{-1} \cdot \frac{v_i(M)}{n} \\ &\geq \; (1-1/n) \cdot MMS_i(n, M), \end{aligned}$$

where the constraint that $v_i(M)/n \geq MMS_i(n, M)$ holds for additive valuations. We show our result for approximate-MMS allocation and our improved bound for approximate-GMMS allocation in section 3.

**1.3. Related work.** Fair division of divisible resources is a classical and well-studied subject starting from the 1940s [32]. Fair division of indivisible goods among competing agents is a young and exciting topic with recent work on EF1 and EFX allocations [15, 9, 30, 11, 14], approximate MMS allocations [13, 12, 2, 8, 28, 26, 23], and approximation algorithms for maximizing Nash social welfare and generalizations [21, 20, 16, 4, 22, 5].

As mentioned earlier, Caragiannis et al. [15] introduced the notion of EFX and it is now known that EFX allocations always exist for three agents with additive valuations [17]. Whether EFX allocations always exist with general valuations or with a larger number of agents is an enigmatic open problem. It was shown in [15] that there always exists an EF1 allocation that is also Pareto-optimal and Barman, Krishnamurthy, and Vaish in [9] showed a pseudopolynomial time algorithm to compute such an allocation.

**Applications.** Fair division of goods or resources occurs in many real-world scenarios and this is demonstrated by the popularity of the website Spliddit (http:

//www.spliddit.org), which implements mechanisms for fair division where users can log in, define what needs to be divided, and enter their valuations. This website guarantees an EF1 allocation that is also Pareto-optimal and since its launch in 2014, it has been used tens of thousands of times [15]. We refer to [27, 30] for details on the diverse applications for which Spliddit has been used: these range from rent division and taxi fare division to credit assignment for an academic paper or group project. Another such website is Fair Outcomes, (http://www.fairoutcomes.com). An interesting application is also *Course Allocate* (used at The Wharton School) which guarantees certain fairness properties to allocate courses among students [30].

**1.4. Improvements with respect to the conference version.** In the conference version of the paper [19], we gave a pseudopolynomial time algorithm to determine an EFX allocation with bounded charity when agents have "gross-substitute" valuations. In the current version we show a pseudopolynomial time algorithm to determine an EFX allocation with bounded charity for all monotone valuations. This is realized with a relaxed definition of "most envious agent" (see Definition 2.3) that helps us to implement the *minimal-envied-subset* oracle efficiently (see subsection 2.1).

## 2. Existence of an EFX-allocation with bounded charity.

We prove our main result on EFX-with-bounded-charity allocations in this section. We will define three *update rules*. Each update rule takes a pair $(X, P)$ consisting of an allocation $X$ and a set $P$ of unallocated goods (we will call $P$ the *pool*) and returns a modified pair $(X', P')$.

Each application of an update rule will ensure that either (i) the *social welfare* $\phi(X) = \sum_{i\in[n]} v_i(X_i)$ of the current allocation increases or (ii) the size of the pool decreases and the social welfare does not decrease, so $|P'| < |P|$ in this case. Hence the update process will terminate. The overall structure of the algorithm is given in Algorithm 2.1.

**Algorithm 2.1** Algorithm for computing an EFX-allocation.

*Postcondition*: $X$ is EFX, $|P| < n$, and $v_i(P) \le v_i(X_i)$ for all $i \in [n]$.

1: $X_i \leftarrow \emptyset$ for $i \in [n]$; $P \leftarrow M$;
2: **while** one of the update rules shown in Algorithm 2.2 is applicable **do**
*Invariant:* $X$ is EFX and the envy-graph $G_X$ is acyclic
3: Let $U_\ell$ be an applicable update rule;
4: $(X, P) \leftarrow U_\ell(X, P)$;
5: Decycle the envy-graph;
6: **end while**

In order to define our update rules, we need the concepts of *envy-graph* and *most envious agent* for a bundle of goods. These were discussed in section 1.2 and we formally define them below.

DEFINITION 2.1. *The* envy-graph $G_X$ *for an allocation* $X = \langle X_1, X_2, \ldots, X_n\rangle$ *has the set of agents as vertices and there is a directed edge from agent $i$ to agent $j$ if and only if* $v_i(X_i) < v_i(X_j)$.

The notion of envy-graph was introduced in [29] and it is well-known that cycles can be removed from the envy-graph without destroying desirable properties (see Lemma 2.2). Thus we can maintain $G_X$ as a DAG.

For each agent $s$, we define the *reachability component* $C(s)$ as the set of all agents reachable from $s$ in the envy-graph. The *sources* of the envy-graph are the vertices with indegree zero. For ease of notation, we will use $B \setminus g$ and $B \cup g$ to denote $B \setminus \{g\}$ and $B \cup \{g\}$, respectively.

LEMMA 2.2. *Let $i_0 \to i_1 \to \cdots \to i_{k-1} \to i_0$ be a cycle in the envy-graph. Consider the allocation $X'$, where $X'_{i_\ell} = X_{i_{\ell+1}}$ (indices are modulo $k$) for $\ell \in \{0, \ldots, k-1\}$ and $X'_j = X_j$ for $j \notin \{i_0, \ldots, i_{k-1}\}$. If $X$ is EFX, then $X'$ is also EFX. Moreover, $\phi(X') > \phi(X)$.*

*Proof.* Consider any agent $i$. We have $v_i(X'_i) \geq v_i(X_i)$ with strict inequality if $i$ lies on the cycle. So $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$. Thus $\phi(X') > \phi(X)$.

Since $X'$ is just a permutation of $X$, for any agent $j$ there exists some agent $j'$ such that $X'_j = X_{j'}$. Therefore, since $X$ is EFX, for any good $g \in X_{j'}$ (or equivalently $X'_j$) we have $v_i(X'_j \setminus g) = v_i(X_{j'} \setminus g) \leq v_i(X_i) \leq v_i(X'_i)$. Thus $X'$ is also EFX. ◻

Let $S \subseteq M$. Suppose there exists an agent who considers $S$ more valuable than her own bundle. Then we will call $S$ an *envied* set. The following definition formalizes the notion of "most envious agent."

DEFINITION 2.3. *Let $Z$ be an* inclusionwise minimal envied subset *of $S$, i.e., (1) there exists an agent $i$ such that $v_i(Z) > v_i(X_i)$ and (2) for all $j \in [n]$ we have $v_j(X_j) \geq v_j(Z')$ for all $Z' \subset Z$. The agent $i$ is called a* most envious agent *of $S$.*

We are now ready to present our three update rules $U_0$, $U_1$, and $U_2$, in Algorithm 2.2.

**Rule $\mathbf{U_0}$.** Rule $U_0$ is the easiest of the update rules. It is applicable whenever adding a good from the pool to some source of $G_X$ does not destroy the EFX-property (see Algorithm 2.2).

LEMMA 2.4 (Rule $U_0$).

(a) *Rule $U_0$ returns an EFX allocation. An application of the rule does not decrease social welfare and decreases the size of the pool.*

(b) *If rule $U_0$ is not applicable, then for any source $i$ of $G_X$ and good $g \in P$, there is an agent $j \neq i$ such that $v_j(X_i \cup g) > v_j(X_j)$. In particular, an inclusionwise minimal envied subset of $X_i \cup g$ has size at most $|X_i|$.*

*Proof.* The first part of (a) follows directly from the precondition of the rule. The second part holds since the valuations are monotone and because $|P'| = |P| - 1$.

The first sentence in part (b) is obvious. We come to the second sentence. Since adding $g$ to $X_i$ destroys the EFX-property, there must be some $g' \in X_i \cup g$ and some $j \in [n]$ such that $v_j(X_i \cup g \setminus g') > v_j(X_j)$ for some $j \in [n]$. Thus an inclusionwise minimal envied subset of $X_i \cup g$ has size at most $|X_i|$. ◻

**Rule $\mathbf{U_1}$.** Rule $U_1$ is applicable whenever there is an agent who values the pool more highly than her current bundle (see Algorithm 2.2).

LEMMA 2.5 (Rule $U_1$). *Rule $U_1$ increases the social welfare and returns an EFX allocation.*

*Proof.* Since there is an agent who values the pool more highly than her own bundle, there is an inclusionwise minimal envied subset $Z$ of $P$ and an agent $i$ such that $v_i(X_i) < v_i(Z)$. Let $X'$ be the allocation defined in Algorithm 2.2, line 7. Then $v_i(X'_i) > v_i(X_i)$ and $v_j(X'_j) = v_j(X_j)$ for $j \neq i$. Thus $\phi(X') > \phi(X)$. It remains to show that the allocation $X'$ is EFX, i.e., for every pair of agents $j$ and $k$ and any good $g \in X'_k$, we have $v_j(X'_k \setminus g) \leq v_j(X'_j)$.

Since $X$ is EFX, this is obvious if neither $j$ nor $k$ is equal to $i$. If $j = i$, then $v_i(X'_i) > v_i(X_i) \geq v_i(X_k \setminus g) = v_i(X'_k \setminus g)$ for all $g \in X'_k$ (or equivalently $g \in X_k$).

**Algorithm 2.2** The update rules.

**function** $U_0$(allocation $X$, pool $P$)
*Precondition:* There is a good $g \in P$ and an agent $i$ such that allocating $g$ to $i$ results in an EFX allocation.
  Allocate $g$ to $i$, i.e., $X'_i \leftarrow X_i \cup g$, $P' \leftarrow P \setminus g$, and $X'_j = X_j$ for $j \neq i$.
  **return** $(X', P')$.
**end function**

**function** $U_1$(allocation $X$, pool $P$)
*Precondition*: There is an agent $k$ such that $v_k(P) > v_k(X_k)$.
  Let $i$ be a most envious agent of $P$; let $Z$ be an inclusionwise minimal envied subset of $P$ such that $i$ satisfies $v_i(Z) > v_i(X_i)$.
  Set $X'_i = Z$ and $X'_j = X_j$ for $j \neq i$.
  Set $P' = X_i \cup (P \setminus Z)$.
  **return** $(X', P')$.
**end function**

**function** $U_2$(allocation $X$, pool $P$)
*Precondition*: There is an $\ell \geq 1$, distinct goods $g_0, g_1, \ldots, g_{\ell-1}$ in $P$, distinct sources $s_0, s_1, \ldots, s_{\ell-1}$ of $G_X$, and distinct agents $t_0$, $t_1, \ldots, t_{\ell-1}$ such that $t_i$ is a most envious agent of $X_{s_i} \cup g_i$ and $t_i \in C(s_{i+1})$ for $0 \leq i \leq \ell - 1$ (indices are to be interpreted modulo $\ell$).
  Let $Z_i$ be an inclusionwise minimal envied subset of $X_{s_i} \cup g_i$ such that $v_{t_i}(Z_i) > v_{t_i}(X_{t_i})$ for $0 \leq i \leq \ell - 1$.
  Set $P' = \left(P \setminus \cup_{i=0}^{\ell-1}\{g_i\}\right) \bigcup_{i=0}^{\ell-1} ((X_{s_i} \cup g_i) \setminus Z_i)$.
  Let $u_0^i \to \cdots \to u_{m_i}^i$ be the path of length $m_i$ from $s_i = u_0^i$ to $t_{i-1} = u_{m_i}^i$ in $C(s_i)$ for $0 \leq i \leq \ell - 1$.
  Set $X'_{u_k^i} = X_{u_{k+1}^i}$ for all $k \in \{0, \ldots, m_i - 1\}$ and all $i \in \{0, \ldots, \ell - 1\}$.
  Set $X'_{t_i} = Z_i$ for all $i \in \{0, \ldots, \ell - 1\}$.
  Set $X'_j = X_j$ for all other $j$.
  **return** $(X', P')$.
**end function**

Finally, we consider $k = i$. Since $Z$ is an inclusionwise minimal envied subset of $P$, we have $v_j(X'_j) = v_j(X_j) \geq v_j(Z \setminus g) = v_j(X'_i \setminus g)$ for any $g \in Z$, where $v_j(X_j) \geq v_j(Z \setminus g)$ follows from the inclusionwise minimality of $Z$ as an envied set. ∎

**Rule $\mathbf{U_2}$.** Rule $U_2$ is our most complex rule. It is applicable if for some $\ell \geq 1$, there are distinct goods $g_0, g_1, \ldots, g_{\ell-1}$ in $P$, distinct sources $s_0, s_1, \ldots, s_{\ell-1}$ of $G_X$, and distinct agents $t_0, t_1, \ldots, t_{\ell-1}$ such that for each $i$, (1) $t_i$ is a most envious agent of $X_{s_i} \cup g_i$ and (2) $t_i$ is reachable from $s_{i+1}$ (indices are to be interpreted modulo $\ell$). We first show that rule $U_2$ is always applicable when rule $U_0$ is not applicable and the pool contains at least $n$ goods.

LEMMA 2.6. *If $|P| \geq n$ and rule $U_0$ is not applicable, then there is an $\ell \geq 1$, distinct goods $g_0, g_1, \ldots, g_{\ell-1}$ in $P$, distinct sources $s_0, s_1, \ldots, s_{\ell-1}$ of $G_X$, and distinct agents $t_0, t_1, \ldots, t_{\ell-1}$ such that $t_i$ is a most envious agent of $X_{s_i} \cup g_i$ and $t_i \in C(s_{i+1})$ for $i \in \{0, \ldots, \ell - 1\}$ (indices are modulo $\ell$).*

*Proof.* Since rule $U_0$ is not applicable, for every source $s$ of $G_X$ and every good $g \in P$, we have $v_j(S') > v_j(X_j)$ for some $S' \subset X_s \cup g$ and $j \in [n]$. Let $s_0$ be an

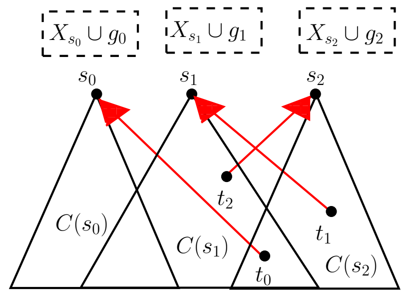


FIG. 3. *We have $t_i$ as a most envious agent of $X_{s_i} \cup g_i$. Recall that $C(s_i)$ is the set of all agents reachable from $s$ in the envy-graph. Neither $t_0 \in C(s_0)$ nor $t_1 \in C(s_0) \cup C(s_1)$. We have $t_2 \in C(s_0) \cup C(s_1) \cup C(s_2)$. Note that $j = 1$ is the largest index in $\{0, 1, 2\}$ such that $t_2 \in C(s_j)$. The cycle (see the proof of Lemma* 2.6) is defined by $s_1, s_2, g_1, g_2, t_1$, and $t_2$.

arbitrary source of $G_X$ and $g_0$ be an arbitrary good in $P$. Construct a sequence of triples $(s_{i+1}, g_{i+1}, t_i)$, $i \geq 0$ defined as follows. Assume we have defined $s_i$ and $g_i$. Let $t_i$ be a most envious agent of $X_{s_i} \cup g_i$. If $t_i \notin C(s_0) \cup \cdots \cup C(s_i)$, let $s_{i+1}$ be a source such that $t_i \in C(s_{i+1})$ and let $g_{i+1}$ be a good in $P$ distinct from $g_0$ to $g_i$. If $t_i \in C(s_0) \cup \cdots \cup C(s_i)$, then stop the construction of the sequence and let $j$ be the maximum index in $\{0, \ldots, i\}$ such that $t_i \in C(s_j)$. Set $\ell = i - j + 1$ and return $s_j, \ldots, s_i$, $g_j, \ldots, g_i$ and $t_j, \ldots, t_i$;[8] see Figure 3 for an illustration.

The construction is well-defined since $|P| \geq n$ and hence we cannot run out of goods. The sources and goods are pairwise distinct by construction. The agents $t_0$ to $t_{i-1}$ are also distinct by construction. Finally, agent $t_i$ is distinct from any agent $t_k$ for $j \leq k < i$ since $t_i \in C(s_j)$ and by definition, $t_k \notin C(s_0) \cup \cdots \cup C(s_k)$ and so $t_k \notin C(s_j)$. ∎

For each $i$, let $u_0^i \to u_1^i \to \cdots \to u_{m_i}^i$ be the path of length $m_i$ from $s_i = u_0^i$ to $t_{i-1} = u_{m_i}^i$ in $C(s_i)$. Rule $U_2$ assigns (i) $X'_{u_k^i} = X_{u_{k+1}^i}$ for all $k \in \{0, \ldots, m_i - 1\}$ and all $i \in \{0, \ldots, \ell - 1\}$ and (ii) $X'_{t_i} = Z_i$ for all $i \in \{0, \ldots, \ell - 1\}$, where $Z_i$ is defined in Algorithm 2.2 (see line 12). For all other $j$, we have $X'_j = X_j$.

LEMMA 2.7 (Rule $U_2$). *Rule $U_2$ increases social welfare and returns an EFX allocation.*

*Proof.* We first observe that the valuations of the agents for their bundles have either increased or remained the same (since either the agents are left with their old bundles or assigned bundles that they envied). In particular, the valuations of all the agents in $\bigcup_{i=0}^{\ell-1} \bigcup_{k=0}^{m_i} \{u_k^i\}$ for their bundles are strictly larger, where the vertices $u_k^i$ are defined above. Thus $\phi(X') > \phi(X)$.

It remains to show that the allocation $X'$ is EFX, i.e., for every pair of agents $j$ and $k$ and any good $g \in X'_k$ we have $v_j(X'_k \setminus g) \leq v_j(X'_j)$. Let $T = \{t_0, t_1, \ldots, t_{\ell-1}\}$. For every agent $k \notin T$ we have $X'_k = X_{k'}$ for some $k'$. Now consider two cases depending on $k$:

- $k \notin T$: Note that the agents' valuation for their current bundle (in $X'$) is at least as good as their valuation for their old bundle (in $X$). We have $v_j(X'_j) \geq v_j(X_j) \geq v_j(X_{k'} \setminus g) = v_j(X'_k \setminus g)$ for any $g \in X'_k$ (or equivalently, $g \in X_{k'}$).

[8]We took $j$ to be the maximum index such that $t_i \in C(s_j)$ so that the cycle defined by $s_j, \ldots, s_i$, $g_j, \ldots, g_i$ and $t_j, \ldots, t_i$ is simple.

- $k \in T$: Let $k = t_i$ for some $i$. We have $v_j(X'_j) \geq v_j(X_j) \geq v_j(Z_i \setminus g)$ for any $g \in Z_i$ (by the inclusionwise minimality of $Z_i$ as an envied set) and $v_j(Z_i \setminus g) = v_j(X'_{t_i} \setminus g) = v_j(X'_k \setminus g)$ for any $g \in X'_k$. ◻

We can now summarize. Let $V = \max_i v_i(M)$ be the maximum valuation of any agent and $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$ be the minimum difference between distinct valuations. Each application of rule $U_1$ or rule $U_2$ increases the social welfare by at least $\Delta$ and hence there can be at most $nV/\Delta$ applications of these rules. Each application of rule $U_0$ decreases the size of the pool by one and so there can be at most $m-1$ successive applications of rule $U_0$ at any intermediate stage in the algorithm. Hence we can conclude that the number of iterations of Algorithm 2.1 is at most $mnV/\Delta + m$. Thus we have shown the following theorem.

THEOREM 2.8. *For monotone valuations, there is always an allocation $X$ and a pool $P$ of unallocated goods such that*
- *$X$ is EFX,*
- *$v_i(X_i) \geq v_i(P)$ for all agents $i$, and*
- *$|P|$ is less than the number of sources in the envy-graph; in particular, we have $|P| < n$.*

*Algorithm* 2.1 *determines such an allocation in at most $mnV/\Delta + m$ iterations.*

**2.1. Finding an inclusionwise minimal envied subset.** We now describe how to implement a *minimal-envied-subset* oracle efficiently, i.e., given an envied set $S \subseteq M$, we need to efficiently find a subset $Z$ of $S$ that is envied by some agent and no proper subset of $Z$ is envied by any agent. Algorithm 2.3 finds such a set $Z \subseteq S$ for monotone valuations.

**Algorithm 2.3** Algorithm for finding an inclusionwise minimal envied subset.

*Precondition*: There is an agent who values $S \subseteq M$ more than her bundle.
*Postcondition*: $Z \subseteq S$ such that there is some agent $i$ with $v_i(Z) > v_i(X_i)$ and we have $v_j(X_j) \geq v_j(Z')$ for all $Z' \subset Z$ and $j \in [n]$.

$Z = S$;
**for** every agent $i$ **do**
  **for** every good $g \in Z$ **do**
    **if** $v_i(X_i) < v_i(Z \setminus g)$ **then**
      $Z = Z \setminus g$;
    **end if**
  **end for**
**end for**
Return $Z$.

In Algorithm 2.3, if the "if condition" is not invoked for any agent $i$ and good $g \in Z$, then $Z$ is an inclusionwise minimal envied subset. Suppose agent $i$ envies $Z \setminus g$ for some $g \in Z$. Then we continue our check with $Z \setminus g$ as the new $Z$. Since the new $Z$ is a subset of the old $Z$, observe that for each agent $i \in [n]$ and each good $g \in Z$, our algorithm needs to check exactly once if $v_i(X_i) \geq v_i(Z \setminus g)$ or not. Thus our algorithm makes $nm$ value queries. We can conclude the following theorem.

THEOREM 2.9. *For monotone valuations, we can determine an allocation $X$ and a pool of unallocated goods $P$ that satisfies the three conditions in Theorem* 2.8 *using at most $mn \cdot (mnV/\Delta + m)$ value queries.*

**2.2. An FPTAS to determine an "almost" desired allocation.** Algorithm 2.1 is pseudopolynomial, since the increase in individual valuations of the agents when we perform the update rules could be very small. Suppose we just wanted an "almost" EFX property, i.e., given any $\varepsilon > 0$, for every pair of agents $i$ and $j$, we are happy to ensure that $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$ and also $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(P)$ for every $i$. We now show an algorithm that determines such an allocation with $\text{poly}(n, m, 1/\varepsilon, \log(V/\Delta))$ many value queries.

This algorithm is obtained from our previous algorithm by making appropriate changes to the definitions of (i) inclusionwise minimal envied subset, (ii) most envious agent, and (iii) update rules $U_0$, $U_1$, and $U_2$. We now describe these changes.

Given a set $S$ such that there is an agent $k$ with $(1+\varepsilon)\cdot v_k(X_k) < v_k(S)$, we say $Z \subseteq S$ is an inclusionwise minimal envied subset if there is an agent $i$ such that $(1+\varepsilon)\cdot v_i(X_i) < v_i(Z)$ and for all agents $j \in [n]$, we have $(1+\varepsilon)\cdot v_j(X_j) \geq v_j(Z')$ for any $Z' \subset Z$, i.e., for all proper subsets of $Z$. Agent $i$ is called a most envious agent of the set $S$.

Note that an inclusionwise minimal envied subset $Z$ of $S$ and a corresponding most envious agent (as defined above) can be determined using $nm$ value queries by changing the "if" condition in Algorithm 2.3 from checking if $v_i(X_i) < v_i(Z \setminus g)$ to checking if $(1+\varepsilon)\cdot v_i(X_i) < v_i(Z \setminus g)$. We now describe the appropriate changes to the update rules $U_0$, $U_1$, and $U_2$.

- Rule $U_0$: This rule is applicable whenever there is an unallocated good $g$ and a source $s$ in the envy-graph $G_X$ such that adding $g$ to $s$ does not destroy the "almost" EFX property, i.e., for all $j \in [n] \setminus \{s\}$, we have $(1+\varepsilon)\cdot v_j(X_j) \geq v_j((X_s \cup g) \setminus g')$ for all $g' \in X_s \cup g$. Rule $U_0$ allocates $g$ to $s$ (as done in the function $U_0$ in Algorithm 2.2).
- Rule $U_1$: This rule is applicable whenever there is an agent $i$ such that $(1+\varepsilon)\cdot v_i(X_i) < v_i(P)$. Rule $U_1$ behaves exactly the same way as the function $U_1$ in Algorithm 2.2—the only changes are in the modified definitions of a most envious agent and an inclusionwise minimal envied subset.
- Rule $U_2$: The preconditions of rule $U_2$ are exactly same as in Algorithm 2.2, but with the modified definition of a most envious agent. Rule $U_2$ behaves exactly the same way as the function $U_2$ in Algorithm 2.2—the only changes are in the modified definitions of a most envious agent and an inclusionwise minimal envied subset.

*The algorithm.* We now argue that with these changes, we can determine an "almost" EFX allocation with bounded charity using $\text{poly}(m, n, 1/\varepsilon, \log(V/\Delta))$ many value queries. Our algorithm always maintains an allocation where for every pair of agents $i$ and $j$, we have $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$. If any of the three update rules $U_0, U_1, U_2$ is applicable, then we apply that rule and decycle the envy-graph. Observe that application of any of these update rules returns an allocation $X$ such that for every pair of agents $(i, j)$, we have $(1+\varepsilon)\cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$. Thus we maintain the invariant of the algorithm.

*Bounding the number of iterations.* The application of any of these update rules and the subsequent envy-cycle elimination does not decrease the valuation of any agent for her bundle. Moreover, each application $U_1$ or $U_2$ will ensure that at least one agent's valuation for her bundle either increases from 0 to $\Delta$ or by a multiplicative factor of $1+\varepsilon$. So the total number of applications of $U_1$ and $U_2$ is $O(n \cdot \log_{1+\varepsilon}(V/\Delta))$. Since each application of $U_0$ reduces the size of the pool by one, we cannot have more than $m$ consecutive applications of $U_0$. Hence the total number of iterations of the algorithm is $O(mn \cdot \log_{1+\varepsilon}(V/\Delta))$, which is $O((mn/\varepsilon)\cdot \log(V/\Delta))$.

When the preconditions of the update rules are no longer valid, we arrive at our desired allocation. Since each iteration involves $O(mn)$ value queries, we can determine our desired allocation with $O(m^2n^2/\varepsilon \cdot \log(V/\Delta))$ value queries. Thus we can conclude the following theorem.

THEOREM 2.10. *For monotone valuations, given any $\varepsilon > 0$, using $O(m^2n^2/\varepsilon \cdot \log(V/\Delta))$ value queries, we can find an allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ and a pool $P$ of unallocated goods such that*

- *for any pair of agents $i$ and $j$, we have $(1+\varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$,*
- *for any agent $i$, we have $(1+\varepsilon) \cdot v_i(X_i) \geq v_i(P)$, and*
- *$|P| < n$.*[9]

**2.3. New proof of a result from [30] for identical valuations.** For agents with identical (monotone) valuations, it was shown by Plaut and Roughgarden [30] that an allocation that maximizes the minimum valuation, then maximizes the size of this bundle, then maximizes the second minimum valuation, then maximizes the size of this bundle, and so on is EFX. Thus a complete allocation that is EFX always exists when all the agents have identical valuations.

We now show that Algorithm 2.1 gives another proof of the above result. Recall that Algorithm 2.1 consists of applying three update rules $U_0, U_1, U_2$—whichever of these is applicable. We will now show that when all the agents have identical valuations and rule $U_0$ is not applicable, then the precondition of rule $U_2$ is satisfied as long as there is some unallocated good. Let $X = \langle X_1, \ldots, X_n \rangle$ be the current allocation and $P = M \setminus \cup_{i=1}^n X_i$ be the set of unallocated goods in $X$.

LEMMA 2.11. *Let $s$ be any source vertex in the envy-graph $G_X$. If $|P| \geq 1$ and rule $U_0$ is not applicable, then $s$ can be chosen as a most envious agent of $X_s \cup g$ for any $g \in P$.*

*Proof.* Let $g \in P$ and $s$ be any source in the envy-graph $G_X$. Since rule $U_0$ is not applicable, there is an agent $t$ such that $v(X_t) < v(S')$ for some $S' \subset S = X_s \cup g$, where $v$ is the common valuation function of all agents. Let $Z \subset S$ be an inclusionwise minimal envied subset of $S$ such that $v(X_t) < v(Z)$. Since $s$ is a source in $G_X$, we have $v(X_s) \leq v(X_t)$. So $v(X_s) < v(Z)$; thus $s$ can be chosen as a most envious agent of $X_s \cup g$. ∎

Lemma 2.11 implies that while $P \neq \emptyset$, either rule $U_0$ or rule $U_2$ is applicable. Whenever we apply rule $U_2$, we add any good $g$ in $P$ to the bundle of a source $s$ in $G_X$ and determine an inclusionwise minimal envied set $Z \subset X_s \cup g$. Note that $v(Z) > v(X_s)$. We then throw the goods in $(X_s \cup g) \setminus Z$ back into the pool $P$ and set $X_s = Z$. This makes agent $s$ strictly better-off and no agent is worse-off; thus we have made progress. So when Algorithm 2.1 terminates, we have an EFX allocation with $P = \emptyset$. Thus we have a complete allocation $X = \langle X_1, \ldots, X_n \rangle$ that is EFX.

**3. Guarantees on efficiency and other notions of fairness.** In this section we assume that all agents have additive valuations. We show that a minor variant of our algorithm finds an allocation that satisfies all the conditions of Theorem 2.8 along with a good guarantee on Nash social welfare *and* MMS.

[9]Observe that $(1+\varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$ implies that $v_i(X_i) \geq (1-\varepsilon) \cdot v_i(X_j \setminus g)$ for all $g \in X_j$ and $(1+\varepsilon) \cdot v_i(X_i) \geq v_i(P)$ implies that $v_i(X_i) \geq (1-\varepsilon) \cdot v_i(P)$ (as $\frac{1}{1+\varepsilon} \geq 1-\varepsilon$). Therefore, Theorem 2.10 implies that we can determine a $(1-\varepsilon)$-EFX allocation with bounded charity with $O(\frac{n^2m^2}{\varepsilon} \cdot \log(\frac{V}{\Delta}))$ value queries.

**Guarantee of high Nash social welfare.** We claimed in section 1 that for additive valuations, it can also be ensured that for each $i$, we have $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$, where $X^* = \langle X_1^*, \ldots, X_n^* \rangle$ is an optimal Nash social welfare allocation and $X$ is the allocation in Theorem 2.8. This is easy to see from Algorithm 2.1:

- Rather than initialize $X_i = \emptyset$, we will initialize $X_i$ to the bundle corresponding to the allocation determined by the algorithm in [14].

So we have $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ to begin with and this is a partial EFX allocation. As the algorithm progresses, our invariant is that $v_i(X_i)$ never decreases for any $i$. So if $X' = \langle X'_1, \ldots, X'_n \rangle$ is the final allocation computed by our algorithm, then we have $v_i(X'_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ for all $i \in [n]$.

PROPOSITION 3.1. *Given a set $N$ of agents with additive valuations and a set $M$ of goods, there exists an allocation $X = \langle X_1, \ldots, X_n \rangle$ and a pool $P$ of unallocated goods that satisfy all the conditions of Theorem 2.8 and $v_i(X_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where $X^* = \langle X_1^*, \ldots, X_n^* \rangle$ is an optimal Nash social welfare allocation.*

**3.1. An approximate MMS allocation if $P$ is large.** We now show that if $|P|$ (the number of unallocated goods in our allocation) is sufficiently large, then our EFX allocation $X$ has a very good MMS guarantee. Recall that our algorithm continues till $|P|$ is smaller than the number of sources in the envy-graph $G_X$ and recall that sources are *unenvied* agents. In particular, if $|P| = n-1$, then the number of sources in $G_X$ is $n$, so no agent envies another. That is, for each $i$, we have $v_i(X_i) \geq v_i(X_j)$ for all $j \in [n]$. Moreover, $v_i(X_i) \geq v_i(P)$. So we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M)}{n+1} \\ &= \left(1 + \frac{1}{n}\right)^{-1} \cdot \frac{v_i(M)}{n} \\ &\geq \left(1 + \frac{1}{n}\right)^{-1} \cdot MMS_i(n, M), \end{aligned}$$

where for every agent $i$, the inequality $MMS_i(n, M) \leq v_i(M)/n$ holds for additive valuations. We formalize the above intuition in Theorem 3.3. The following proposition from [23] will be useful. It states that if we exclude any set of agents and at most the same number of any goods from $N$ and $M$, respectively, the MMS of any remaining agent can only increase.

PROPOSITION 3.2 ([23]). *Let $N$ be a set of $n$ agents with additive valuations and $M$ be a set of $m$ goods. If $N' \subseteq N$ and $M' \subseteq M$ are such that $|N \setminus N'| \geq |M \setminus M'|$, then for any agent $i \in N'$, we have $MMS_i(n', M') \geq MMS_i(n, M)$, where $n' = |N'|$.*

THEOREM 3.3. *Given a set $N$ of $n$ agents with additive valuations and a set $M$ of $m$ goods, there exists an allocation $X = \langle X_1, \ldots, X_n \rangle$ and set $P$ of unallocated goods that satisfies*

- *the three conditions stated in Theorem 2.8;*
- *$v_i(X_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where $X^*$ is an optimal Nash social welfare allocation;*
- *$v_i(X_i) \geq MMS_i(n, M) / \left(2 - \frac{k}{n}\right)$ for every $i \in N$, where $k = |P|$.*

*Proof.* Let $(X, P)$ be the allocation guaranteed by Proposition 3.1. Hence the first two conditions given in the theorem statement are satisfied by $(X, P)$. So what we need to show now is that for any agent $i$, we have $v_i(X_i) \geq MMS_i(n, M) / \left(2 - \frac{k}{n}\right)$.

We fix some agent $i$ and let $N' \subseteq N$ be the set of agents consisting of all sources of $G_X$, agent $i$, and all other agents $j$ with $|X_j| \geq 2$. Let $M'$ be the set of goods allocated to the agents in $N'$. Observe that every agent in $N \setminus N'$ is allocated at most one good and so $|N \setminus N'| \geq |M \setminus (M' \cup P)|$. By Proposition 3.2, it holds that $MMS_i(n', M' \cup P) \geq MMS_i(n, M)$, where $n' = |N'|$. Thus, it suffices to show that $v_i(X_i) \geq MMS_i(n', M' \cup P)/\left(2 - \frac{k}{n}\right)$.

Consider any agent $j \in N'$ with $|X_j| \geq 2$. Because $X$ is EFX, it holds that $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$. Since the valuations are additive, we have

$$v_i(X_i) \;\geq\; \left(1 - \frac{1}{|X_j|}\right) \cdot v_i(X_j) \;\geq\; \frac{1}{2} \cdot v_i(X_j).$$

We know the following inequalities hold:

$$v_i(X_i) \geq v_i(P), \tag{3.1}$$

$$v_i(X_i) \geq v_i(X_j) \text{ for all } j \text{ that were sources in } G_X, \tag{3.2}$$

$$2v_i(X_i) \geq v_i(X_j) \text{ for all other } j \in N'. \tag{3.3}$$

Recall that the number of sources is at least $|P| + 1 = k + 1$. Summing up all inequalities in (3.1)–(3.3) and using the fact that $v_i$ is additive, we have

$$(2(n' - (k+1)) + k + 2) \cdot v_i(X_i) \;\geq\; v_i(M' \cup P).$$

Hence we have

$$\begin{aligned}
v_i(X_i) \;&\geq\; \frac{v_i(M' \cup P)}{2n' - k} \\
&= \frac{v_i(M' \cup P)}{n'} \cdot \frac{n'}{2n' - k} \\
&\geq\; MMS_i(n', M' \cup P) \cdot \frac{n'}{2n' - k} \quad \text{(since } v_i \text{ is additive)} \\
&=\; MMS_i(n', M' \cup P)/\left(2 - \tfrac{k}{n'}\right) \\
&\geq\; MMS_i(n', M' \cup P)/\left(2 - \tfrac{k}{n}\right) \quad \text{(since } n' \leq n\text{)}.
\end{aligned}$$

Thus $v_i(X_i) \geq MMS_i(n, M)/\left(2 - \frac{k}{n}\right)$ for every $i \in N$, where $k = |P|$. ∎

**3.2. An improved bound for approximate-GMMS.** As mentioned in section 1, a new notion of fairness called GMMS was recently introduced by Barman et al. [7]. We formally define a GMMS allocation below.

DEFINITION 3.4. *Given a set $N$ of $n$ agents and a set $M$ of $m$ goods, an allocation $X = \langle X_1, X_2, \ldots, X_n \rangle$ is $\alpha$-GMMS if for every $N' \subseteq N$ and all $i \in N'$, we have*

$$v_i(X_i) \;\geq\; \alpha \cdot MMS_i(n', \bigcup_{j \in N'} X_j), \text{ where } n' = |N'|.$$

Observe that a GMMS allocation is also an MMS allocation. Since MMS allocations do not always exist in a given instance [28], GMMS allocations also need not always exist. Interestingly, $\frac{1}{2}$-GMMS allocations always exist [7]. We now describe how to modify our allocation to result in a complete allocation that is $\frac{4}{7}$-GMMS and also $\frac{1}{2}$-EFX. An allocation $Y = \langle Y_1, \ldots, Y_n \rangle$ is $\frac{1}{2}$-EFX if for any pair of agents $i, j$, we have $v_i(Y_i) \geq \frac{1}{2} \cdot v_i(Y_j \setminus \{g\})$ for any $g \in Y_j$. Plaut and Roughgarden [30] showed that when all agents have subadditive valuations, a $\frac{1}{2}$-EFX allocation always exists. Theorem 2.8 along with the simple modification of allocation $X$ to allocation $Y$ described below gives an alternative proof that a $\frac{1}{2}$-EFX allocation always exists for subadditive valuations.

Let $X = \langle X_1, \ldots, X_n \rangle$ be the allocation and $P$ be the pool of unallocated goods that satisfy the conditions of Proposition 3.1. Without loss of generality, assume that agent 1 is a source in the envy-graph $G_X$. Define the complete allocation $Y = \langle Y_1, \ldots, Y_n \rangle$ as follows:

* $Y_1 = X_1 \cup P$ and $Y_i = X_i$ for all $i \neq 1$.

Theorem 3.5 shows that $Y$ is our desired allocation. The proof of Theorem 3.5 is similar to [1, Proposition 3.4]. We also remark that one can use the proof of [1, Proposition 3.4] to show that any EFX allocation is 4/7-GMMS. However, note that $Y$ is not necessarily an EFX allocation. But it has sufficiently nice properties so that we can still show that it is 4/7-GMMS. For the sake of convenience, we will refer to goods in the proof of Theorem 3.5 as *items*.

THEOREM 3.5. *Given a set $N$ of $n$ agents with additive valuations and a set $M$ of $m$ items, the allocation $Y$ defined above satisfies the following:*

- *$Y$ is $\frac{4}{7}$-GMMS and $\frac{1}{2}$-EFX.*
- *$v_i(Y_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where $X^*$ is the optimal Nash social welfare allocation.*[10]

*Proof.* Observe that the bound on Nash social welfare holds for allocation $X$ and thus for allocation $Y$ (since $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$). The proof that $Y$ is $\frac{1}{2}$-EFX is straightforward. Recall that $Y_1 = X_1 \cup P$ and $v_i(X_i) \geq v_i(X_1)$ and $v_i(X_i) \geq v_j(P)$ for all $i \in [n]$. Thus for any subadditive valuation $v_i$, we have $v_i(Y_1) \leq v_i(X_1) + v_i(P) \leq 2v_i(Y_i)$. For $j \neq 1$, we have $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus g) = v_i(Y_j \setminus g)$ for all $g \in Y_j$. Hence $Y$ is $\frac{1}{2}$-EFX.

What we need to show now is the guarantee on GMMS. That is, we need to show that for every $\widetilde{N} \subseteq N$ and all $i \in \widetilde{N}$, we have $v_i(Y_i) \geq \frac{4}{7} MMS_i(\widetilde{n}, \widetilde{M})$, where $\widetilde{n} = |\widetilde{N}|$ and $\widetilde{M} = \bigcup_{j \in \widetilde{N}} Y_j$.

Fix some $i \in \widetilde{N}$. Define $N'$ to be the subset of $\widetilde{N}$ that contains $i$ and all agents that have been allocated at least two items in $Y$, i.e., $j \in N'$ if and only if $j = i$ or $|Y_j| \geq 2$. Let $M' = \bigcup_{j \in N'} Y_j$.

Note that $Y$ allocates all items of $\widetilde{M} \setminus M'$ to agents in $\widetilde{N} \setminus N'$. Since every agent in $\widetilde{N} \setminus N'$ has been allocated at most one item, we have $|\widetilde{N} \setminus N'| \geq |\widetilde{M} \setminus M'|$. Proposition 3.2 tells us that $MMS_i(n', M') \geq MMS_i(\widetilde{n}, \widetilde{M})$, where $n' = |N'|$. Thus it suffices to show $v_i(Y_i) \geq 4/7 \cdot MMS_i(n', M')$.

For $j \in N'$, call a bundle $Y_j$ "good" if $j \in \{1, i\}$ or $|Y_j| = 3$; call it "bad" otherwise. Thus $Y_j$ is bad if $j \notin \{1, i\}$ and $|Y_j| = 2$. Call items in good bundles "good" and items in bad bundles "bad." Let $x$ be the number of bad items in $M'$. Since the bad items are contained in bundles of size two, the good items in $M'$ come from $n' - x/2$ good bundles of $Y$. As long as $x > n'$, we will apply a reduction step. Each reduction step will reduce the number of bad items in $M'$ by two and the number of agents by one, will not decrease the $MMS_i$-value, and will leave the quantity $n' - x/2$ and set of good items in $M'$ unchanged.

[10]Related results were independently obtained by Garg and Taki in [24], where it was shown that for *additive* valuations, there is an EFX-allocation after donating at most $n-1$ items to charity. However, no bound on the value of the items donated to charity was shown. Thus a 4/7-GMMS-allocation after removing $n-1$ items from the original set of items was obtained by them.

Let $Z = \langle Z_1, Z_2, \dots Z_{n'} \rangle$ be an optimal MMS partition for agent $i$ of the set $M'$ of items. If there are more than $n'$ bad items in $M'$, then there is a set $Z_k$ with at least *two* bad items, say, $g_1$ and $g_2$. We distribute the items in $Z_k \setminus \{g_1, g_2\}$ arbitrarily among the other sets in $Z$. So we have a partition of the set $M' \setminus \{g_1, g_2\}$ of items into $n' - 1$ many bundles. The value for agent $i$ of any remaining bundle did not decrease. We set $M'$ to $M' \setminus \{g_1, g_2\}$ and decrement $n'$. Note that we reduced the number of bad items by two, the number of agents by one, did not decrease $MMS_i(n', M')$, and the good items in $M'$ still come from the $n' - x/2$ good bundles in $Y$. We keep repeating this reduction until $M'$ contains at most $n'$ bad items. At this point, we have a set $M'$ of items and an integer $n'$ with the following properties:

1. The number $x$ of bad items in $M'$ is at most $n'$.
2. $MMS_i(n', M') \geq MMS_i(\widetilde{n}, \widetilde{M})$.
3. The set of good items in $M'$ has not changed. The good items come from $n' - x/2$ good bundles in $Y$.

We will next relate the value of good and bad items to the value of $Y_i$.

Claim 3.6. *We have the following:*
(a) *For any bad item $g$, $v_i(g) \leq v_i(Y_i)$.*
(b) *If $j \neq 1$ and $Y_j$ is a good bundle, then $v_i(Y_j) \leq 3/2 \cdot v_i(Y_i)$.*

The proof of Claim 3.6 is given below. Now we are ready to show the bound on GMMS. We will show that $v_i(Y_i) \geq \frac{4}{7} v_i(M')/n'$. Since $v_i(M')/n' \geq MMS_i(n', M') \geq MMS_i(\widetilde{n}, \widetilde{M})$, we get the desired bound.

We have $x$ bad items in $M'$. The good items in $M'$ come from $n' - x/2$ good bundles. Also $x \leq n'$. The total value of the good items for agent $i$ is at most

$$\frac{3}{2}\left(n' - \frac{x}{2} - 2\right) \cdot v_i(Y_i) + 2v_i(Y_i) + v_i(Y_i) = \frac{3}{2}\left(n' - \frac{x}{2}\right) \cdot v_i(Y_i).$$

This is because either (i) $1 \in N'$ and then there are $n' - x/2 - 2$ good bundles different from $Y_1$ and $Y_i$ or (ii) $1 \notin N'$ and so there are $n' - x/2 - 1$ good bundles different from $Y_i$. We know from Claim 3.6 that each good bundle $Y_j$, where $j \neq 1$, has value at most $3/2 \cdot v_i(Y_i)$ and we showed earlier that $Y_1$ has value at most $2v_i(Y_i)$.

Also, the total value of the bad items for agent $i$ is at most $x \cdot v_i(Y_i)$, since there are $x$ many bad items and each bad item is worth at most $v_i(Y_i)$ (by Claim 3.6). Therefore,

$$\begin{aligned} v_i(M') &= v_i(\text{bad items in } M') + v_i(\text{good items in } M') \\ &\leq x \cdot v_i(Y_i) + \frac{3}{2}\left(n' - \frac{x}{2}\right) \cdot v_i(Y_i) \\ &= \frac{6n' + x}{4} \cdot v_i(Y_i) \\ &\leq \frac{7n'}{4} \cdot v_i(Y_i). \end{aligned}$$

∎

*Proof of Claim* 3.6. We prove (a) and (b) below.
(a) Let $Y_j = \{g, g'\}$ be the bundle containing $g$. Since $j \neq 1$ (by the definition of a bad item), $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus g') = v_i(Y_j \setminus g') = v_i(g)$.

(b) Let $g \in Y_j$ be such that $v_i(g)$ is minimal. Then $v_i(Y_j \setminus g) \leq v_i(Y_i)$ and $v_i(g) \leq v_i(Y_j)/|Y_j|$. Thus

$$v_i(Y_i) \geq \left(1 - \frac{1}{|Y_j|}\right) \cdot v_i(Y_j) \geq \left(1 - \frac{1}{3}\right) \cdot v_i(Y_j) = \frac{2}{3} \cdot v_i(Y_j).$$

This finishes the proof of Claim 3.6. ∎

## 4. Conclusions and open problems.

We studied the existence of EFX allocations when agents have monotone valuations. We showed that we can ensure such an allocation always exists when we donate a small number of goods that nobody envies to charity. The major open problem here is whether complete EFX allocations always exist. Plaut and Roughgarden [30] remarked that an instance with *no* complete EFX allocation may be easier to find in the general setting of monotone valuations. Our result on "almost-EFX" allocations for monotone valuations allows one to hope that complete EFX allocations always exist, at least for more structured valuations such as additive.

We also showed that we get guarantees in terms of other notions of fairness when agents have additive valuations. To the best of our knowledge, allocations with good guarantees (i.e., constant factor approximation) on Nash social welfare and MMS (as well as GMMS) were not known prior to our work. It would also be interesting to investigate whether these guarantees can be improved or if instances can be constructed where our guarantees are tight. We believe that our work is just the beginning toward determining an allocation that gives good guarantees with respect to several notions of fairness: an allocation that is *universally* fair.

Very recently, Amanatidis, Ntokos, and Markakis [3] have announced an allocation that has good approximation guarantees simultaneously with respect to four notions of fairness when valuation functions are additive: in particular, their allocation is $(\phi - 1)$-EFX and $2/(\phi + 2)$-GMMS, where $\phi \approx 1.618$ is the golden ratio. Moreover, a fine-tuned version of their algorithm also achieves 4/7-GMMS, which matches our result (see Theorem 3.5). They also show that for additive valuations, when $m$ is at most $n + 2$, where $m$ is the number of goods and $n$ is the number of agents, GMMS (and hence, EFX) allocations always exist. Even more recently, Berger et al. [10] showed that when agents have additive valuations, an EFX allocation $(X, P)$ where $|P| \leq n - 2$ always exists; moreover, when $n = 4$, $|P| \leq 1$, i.e., at most one good is unallocated.

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